Fuzzy Set Theory Movement in the Social Sciences

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Does fuzzy set theory apply to public administration? Or more aptly, when does it not apply to public administration? William A. Treadwell's exploratory article moves us toward an understanding of the "fuzzy" world in which we work. A fuzzy set theory orientation can bring together disparate theoretical interpretations and allow them to mutually co-exist as they are each allowed to provide partial explanation of the phenomenon at hand. Treadwell proposes a "fuzzy acceptance theory", with which a framework is provided for pulling together different theoretical perspectives that allows each perspective the ability to provide partial explanatory power for the issue at hand.

Some very familiar constructs in the realm of public administration discourse are supervisor-worker, teacher-student, public-private, democracy-autocracy, conservatism-liberalism, patronage-merit, individualism-collectivism, and centralized-decentralized. These and other constructs and theories in public management are bantered about through iterations of paired comparisons of contrasting relationships that are framed in a language of dichotomous symbolism. Even when a construct does not appear to lend itself readily to a bi-polar structure, it will implicitly exist, as in the familiar constructs of representation, efficiency, effectiveness, equity, and property; in these examples the negative of the terms form the bi-polar comparison point—no representation, no efficiency, no effectiveness, no equity, no property.

Our temporal process is uniquely bi-polar (Kelly 1955; Adams-Webber 1979; Rychlak 1981), where our constructs tend to be labeled by their similarity pole. When we speak of what something means, we are always referring to a relationship between similarities and differences. Kelly's personal construct theory is based on the notion of a construing process where thought is only possible because humans dichotomize experience into similarities and contrasts, as our bi-polar nature. Personal construct theory can be extended into an ontological case for fuzziness as presented by Kosko (1989) wherein: The universe consists of all subsets of the universe. The only subsets of the universe that are not fuzzy are the constructs of classical mathematics. All other sets—sets of particles, cells, tissues, people, ideas, galaxies—in principle contain elements to different degrees. Their membership is partial, graded, inexact, ambiguous, or uncertain. To the extent that the human construes reality as sets of bi-polar constructs, the strength of any construct would then be based on its membership gradient value at the time an event occurs. The membership gradient of any construct is a matter of perceived similarities and contrasts. At this point, it may be best to stop and back up, because we have the cart before the horse. Presenting an historical introduction to logic and fuzzy theory will better orient our frame of reference.

Currently in the United States, people who are over 30 years of age were traditionally raised using a nonmetric system of weights and measures—using terms such as feet, ounces, and quarts. The younger generations have been raised using the metric system of weights and measures of meters, grams, and liters. The transition of the older generation to using the metric system has not been very successful. The old ways are still thoroughly embedded in society. There is a split between the conflicting institutionalized old way and the new. The same may hold true in the coming years between the use of classical set theory and the rise in the use of fuzzy set theory. Classical set theory has a long documented history starting with Aristotle; fuzzy theory is a mere child of the 1960s.

The dialogue between the human sciences and fuzzy set theory has been scattered, unsystematic, and slow to develop. According to Smithson (1987) "...fuzzy set mathematics are couched in foreign and rather obtuse notation which is forbidding even to the more mathematically sophisticated behavioral scientist." Virtually all texts on the topic assume either a mathematical or computer science and engineering orientation. Smithson's book was an attempt to bridge the gap and lighten the burden of the mathematics to illustrate the basic elements of fuzzy set theory in real-world research examples taken from cognitive psychology, social psychology, sociology, social anthropology, political science, and evaluation research.

In the same spirit of Smithson, the goal of this expository article on fuzzy theory is to minimize the obtrusion of mathematics and make the topic more palatable to a wider audience. This initial paper sets out to provide some structure for handling fuzzy concepts and illustrate their use in budgeting and decision making.

What distinguishes fuzzy set theory from classical set theory? According to classical set theory, an element either belongs to one set or it belongs to another. In fuzzy set theory an element may belong partially to a set. Fuzzy sets have gradations of set membership and blurred boundaries. Classical theory has well defined set boundaries and membership is as clear as black and white. At issue is to which set does gray belong? Classical theorist may attempt to create a new set called gray; yet the problem still persists. When does dark gray become black or conversely when does light gray become white? The fuzzy-theory approach neatly handles the assignment of gray as a partial member of both the white and black sets. The darker the gray, the more it tends to be a member of the set black and less a member of the white set. The world of perception does not have sharp edges. It is full of ambiguity and uncertainty, and it is only reasonable then to promote the use of fuzzy membership assignments. One can begin to see that classical sets, using mutual exclusivity as the defining operative, create boundaries that are actually the zones over which conflict reigns.

The Development of Formal Logic

It is important to understand the principles of logic in order to grasp what rational thought is. For one, it is not natural. Rational thought is learned, then applied. There are many systems of logic that have been developed in support of rational thought. In logical knowledge representations, facts, knowledge, and rules are expressed in terms of predicates and logical sentences. Parsaye and Chignell (1988) commented "formal logic has been developed in spite of inherent tendencies of humans toward irrational and emotional behavior often founded on anything but logic." Part of the attraction of formal logic is that it acts as a counterweight to human irrationality. Parsaye and Chignell suggested that that is why the formal mathematical ideals behind the economic thoughts and formulations of supply and demand in a free market system still have appeal to economists.

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Aristotle is generally considered to be the founder of logic. The ancient Greeks used oratory to put forward or refute arguments in their process of public debate. Aristotle developed a method called syllogistic logic for analyzing and evaluating arguments. An example of syllogistic logic, is the following syllogism: All men are mortal; Aristotle is a man; we then deduce Aristotle is mortal. In syllogistic reasoning new information is logically deduced from preceding information. In addition, when Aristotelian logic is used, it implies that everything can be identified as belonging to one category or another. There are no shades of gray; no conception of "partly" or "mostly"—everything exists in mutually exclusive categories.

A century following Aristotle, Chryssipus developed propositional logic as a method to understand compound propositions (sentences) as either true or false depending upon their components. A thousand years later William of Occam developed modal logic during the 14th century. Modal logic includes concepts such as possibility, necessity, belief, and doubt.

In 1854, Boole published *An Investigation of the Laws of Thought on Which are Founded the Mathematical Theories of Logic and Probabilities.* Boole translated the arithmetic operators of addition, multiplication, and subtraction and created their logical set equivalents—the union operation by applying the minimum rule, the intersection operation using the maximum rule, the connective "not" and the truth table.

Boolean logic was later codified into symbolic logic by Russell and Whitehead in *Principia Mathematica*, published in 1917. The foundation of formal logic is often attributed to Russell and Whitehead (Parsaye and Chignell, 1988). When classical logic is applied to a set of variables containing arguments and predicates that create a clause to expose a fact, or when a logical connective is made to form a compound clause, the result is either true or false. There is no allowance for uncertainty within the rules of a formal logic system. Formal logic has many useful features, but it does not explicitly represent uncertainty in reasoning.

The approach to dealing with uncertainty in reasoning has primarily been with the application of Bayes theories of prob-

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abilities. The Bayesian analysis uses a procedure which assigns prior probabilities that an event(s) will occur, then data are collected, and these data are then used to change the prior probabilities, to yield posterior probabilities. This process can be recursively applied to the re-occurring event and continuously modify probabilities of an event(s) future prospect to be realized. The key idea of Bayesian statistics is prior opinions are changed by data to yield posterior opinions (Phillips, 1973). All uncertainties are stated as percentage of likelihood that something will occur. For example, while listening to the radio broadcaster reading the news, he announced..."there is a 10 percent chance of rain today." Then he spontaneously adjusted his script, "and it's 10 percenting right now." Probability does not help in determining to what extent an event is actually occurring, other than stating 100 percent. The measurement of how heavy it is raining is a fuzzy measurement ranging from not at all to a heavy downpour.

The Foundation of Fuzzy Set Theory

Jan Lukasiewicz, a Polish mathematician during the 1920s, developed the basic idea of multivalued logic. Forty years later in 1965, it was expanded upon and called Fuzzy Theory by Lotfi A. Zadeh. Zadeh's first paper, "Fuzzy Sets" (1987), introduces the fuzzy set and the mathematical definitions of inclusion, union, intersection, compliment, relation and convexity as a derivative of Boolean logic (Appendix A). Later work by Zadeh employs fuzzy logic with the purpose of modeling how people reach conclusions when the information available is imprecise, incomplete and not totally reliable. Such modeling is approached by the interpretation of natural language through the representational mechanisms of fuzzy sets and possibility theory.

While the foundations of fuzzy set theory are not debated, there is controversy over the applications of fuzzy theory to real world events. Fuzzy set theory is not the panacea for dealing with the world of uncertainty in certain terms, but it is a strong contender. Smithson (1987) notes:

> The principal value I find in fuzzy set theory is that it generates alternatives to traditional methods and approaches, thereby widening the range of choices available to researchers. The more alternatives we have, the more possible it becomes to conduct high quality research, and the fewer excuses we have for taking default options. The more sophisticated our linguistic and conceptual frameworks, the less likely we are to trivialize or distort our questions in the name of mere tractability.

Fuzzy theory does not use probabilities nor view any situation's uncertainty as due to randomness. Fuzzy theory handles uncertainty as deterministic. Where Bayesian theorists see probabilities, fuzzy theorists see different amounts of membership in events that are not probable but as real events. When one sets out to make a set of inferences or predictive statements, they are predicated on prior descriptions that embellish various levels of certainty regarding occurrence, which are deterministic in their origination. There is nothing random nor probable when you measure such information. As Kosko (1989) has noted:

> This count does not involve randomness. It counts which elements are identical or similar and to what degree. The phenomena themselves are deterministic. The corresponding frequency number that summarizes the deterministic situation is also deterministic. The same situation always gives the same number. The number may be used also to place bets or to switch a phone line, but it remains part of the description of a specific state of affairs. The deterministic subsethood derivation of relative frequency eliminates the need to invoke an undefined 'randomness' to further describe the situation.

The hundred years of effort to building probability theory into a descriptive universe had been challenged by the fuzzy theory approach. After Kosko (1989) presented a new geometric proof of The Fuzzy Subsethood Theorem, he illustrated that fuzzy theory is an extension of probability theory—and in particular, Bayes Theorem. Probability is a special case of fuzziness, where Bayesian theory is subsumed into fuzzy theory as a special subset.

Fuzzy logic allows one to express uncertainty within a rule—inexact reasoning system—such that a fuzzy logic conclusion is not stated as either true or false, but as being possibly true to a certain degree. The degree of certainty is called the "truth value." Fuzzy set theory uses only the numeric interval of 0 to 1:

FALSE: Truth Value = 0

TRUE : Truth Value = 1

UNCERTAIN: 0 < Truth Value < 1

Often dealing with uncertainty, we are not completely sure of a fact, but we have reason to believe that it is "possibly" true. The possibility is one with the term fuzziness used to describe event ambiguity. In contrast to other measures, it measures the degree to which an event occurs, not whether it occurs. Whether an event occurs is "random." To the degree it occurs is fuzzy. Fuzziness depicts these relationships in a sets-as-points perspective. A fuzzy set is a point in a unit hypercube and a nonfuzzy set is a corner of the hypercube. The fuzzier A is, the closer A is to the midpoint in the fuzzy cube. As A approaches a vertex the less fuzzy it is (or more like the vertex than not). The Rubik's cube is a good metaphor for understanding the fuzzy sets-as-points hypercube orientation. The corners of the Rubik's cube represent the defined values of traditional Aristotelian logic, whereas the points inside the cube correspond to fuzzy logic sets. Visualizing this geometry may by itself be the more powerful argument for fuzziness (Appendix B).

Linguistic Variables, Quantifiers, and Hedges

If we remove one stone from a large pile, the pile remains large. Reapplying this concept over and over will ultimately lead to a pile with just one stone. When does a large pile become a small pile? If you start with the largest known organization, and remove one employee at a time from it, when does it become a small organization? Both the pile's transition and the organization's transition occur within the looseness allowed by the vague and inexactness of the word "large." "Large" is a fuzzy concept. The concepts "appearance" and "young" are also examples of fuzzy concepts.

Compatibility is distinct from that of probability. According to Zadeh, compatibility is merely a subjective indication of the ability of one's conception of a label to describe accurately a particular value. Zadeh explained the ideas of fuzzy quantifiers such as "few," "seldom," "usually," "often," "many," and "rarely," to imply a fuzzy numeric value, such as "I often drive that route" or "I frequently drive by myself."

Smithson (1987) conducted research on numeric fuzzy values associated with the fuzzy quantifier "several." He asked 23 students to rate the degree of possibility that various integers could be the number someone has in mind when they say several. His results indicated a fair consensus on the matter. Zetenyi's (1988) review indicated that quantifiers, such as several, are very much affected by context. The expected frequency of the event, the type of activity, the set size described, and the range of alternative quantifiers available can each affect qualifiers.

Hormann (1983) experimentally illustrated that contextual situations influence the amount indicated by a quantifier. For example the quantifier "few" changes in the following contextual situations:

- By object—*a few crumbs* means more than 8 whereas *a few shirts* means approximately 4.
- By size of object—*a few large cars* suggest a smaller number than *a few small cars.*
- By spatial location—*a few people standing in front of a hut* is not as many as *a few people standing in front of a building.*

Hence the range of quantifiers is determined by what other quantifiers are available; therefore the sets indicated by quantifiers must be able to expand and contract according to the context in which they occur. The important regularity Zetenyi noted for building a model of quantifiers is "a complex but coherent conceptualization: quantifiers can be characterized as fuzzy sets located in fixed order along an analogue scale. The size of the set is variable, and some quantifiers are more mobile than others in their position along this scale." He described a model of fuzziness embedded within fuzziness.

Zadeh identified another set of natural language words that he referred to as "linguistic hedges." Examples of some hedge words are "very," "quite," "moderately," "more or less," "somewhat," "rather," and "sort of." Hedge words either increase or decrease linguistic variables. To illustrate hedges, consider the statement, "John is essentially decent." "Decent" is a linguistic variable that can take on components such as "kind," "honest," "polite," and "attractive." The word "essentially" is a linguistic hedge. Analyze the following statement, "bureaucracies are rarely efficient."

If you start with the largest known organization, and remove one employee at a time from it, when does it become a small organization?

Fuzzy Social System Models

Kaufmann and Gupta (1988) believe classical models, which have worked well for simple and isolated natural phenomena, are not necessarily suited to explore contemporary problems with their complexity, interactions, and human subjectivity. Deterministic and probabilistic mathematical tools have been developed in the conventional systems theory that obey the well-defined physical laws such as Newton's laws of motion and gravitation and Ohm's law for electrical circuits in mechanical systems. According to Kaufmann and Gupta, attempts to extend these system-theory models to such systems as biological processes and socio-economic processes, to a large extent, have provided no benefits.

Kaufmann and Gupta (1988) have adapted current system theory models into the context of fuzzy system models. Kaufmann and Gupta's method for zero-base budgeting used a special class of fuzzy numbers known as triangular fuzzy numbers (TFN; see appendix C for more detail). Zero-base budgeting works on the premise that each new year's budget has to be justified on the perceived need for each program within every department, division, and agency at any funding level. As is true with any type of budgeting process, it is difficult to develop a precise budget. For this reason, the use of fuzzy data rather than the deterministic-numeric approach may produce more realistic results.

In Kaufmann and Gupta's model, each decision center in an organization submits one or more budgets based on different operating premises and goals. Every budget is summarily presented in the context of three outcomes: a minimal budget, a normal budget, and an improvement budget. Each budget can then be represented as an interval of confidence with lower and upper thresholds. Kaufmann and Gupta's example used a company with four decision centers that submitted 12 budgets: two centers submitted three budgets, one center provided two budgets and the fourth center presented four budgets. Using fuzzy set theory, it can be illustrated which combinatory set(s) of cumulative budgets can be adopted without restriction or adopted with minimal risk or even high risk to the organization, all within the constraints of the organization's upper threshold of funding.

While the fuzzy zero-base budget model separated sets of cumulative budgets between being acceptable without risk and

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acceptable with different possible levels of risk to the organization, it did not produce information necessary to determine which cumulative budget ultimately to adopt. Which of the six is to be chosen? The Kaufmann and Gupta paper did not move into the process of how one would pick one budget from among the six—such an objective was beyond the scope of their paper. Another set of criteria and operations are needed to provide the final decision information beyond the total dollar amount provided by the budgets. As Kaufmann and Gupta (1988) stated in the introduction of their work, "fuzzy optimization is a process which seeks a good solution without being able to prove that it is the best (optimum). However, in the real world, it is difficult to determine what is meant by the best." Difficult or not, one of the six zero-base budgets must be adopted, which one will it be?

Fuzzy Decision Making

Yager's (1980) technique for multi-objective decisions may be appropriate in determining which fuzzy-derived zero-base budget to accept. Yager's (1980) position was decision makers do not have a set of multi objectives which always meet all of the desired requirements. As such, the situation is not one which a particular set of objectives will satisfy the decision maker completely, although each of the decision sets will provide a certain degree of satisfaction to the decision maker. Yager formulated some ideas on how to evaluate alternatives under these conditions. He illustrated the fuzzy decision process with the example of how to go about selecting a candidate for a job.

Yager employed a two-step process that used both the fuzzy set minimum rule and maximum rule. To chose among candidates, the process first aligned candidates according to their weakest evaluated attribute regardless of the objective criteria. Once the weakest attribute had been exposed among all candidates, the strongest appearing candidate with the least weakness was then selected.

Yager next moved the current decision-making process of selecting a candidate to another level of complexity by stating that the decision conditions were not of equal importance. The addition of relative levels of importance among the criteria translates into a hedge function. Each candidate's evaluation is multiplied by the respective weights to create a weighted matrix. Next the selection process returns to the two step minimum-maximum fuzzy set decision process.

Translating Yager's example into terms for selecting a single zero-base budget, recall where we left in our example: six budgets were found to be within acceptable organizational levels of risk, and they each met the constraints of the upper limits of funding for the organization. Hence the budget to be adopted must be the one that will make the organization successful. Success is derived from the goals or the mission statement of the organization. In this example "success" is the operative word that the fuzzy analysis will hinge upon. The set of multi-objective constructs are the programs within departments of the organization. Each department in the organization is then evaluated on two attributes. First, a variable is assigned based on the department's perceived ability to fulfill its stated objective within its budget, and second, a hedge function of how important each specific department's contribution is to the overall success of the organization. The resulting matrix can then be resolved using the two step minimum-maximum fuzzy set decision process.

Problems with Fuzzy Theory

This process may not always be capable of deciding upon which alternative an organization should select. Therefore the application of the fuzzy decision model may not always lead to a conclusive result.

Kickert's (1978) critical review of fuzzy decision making, concluded two problems exist in fuzzy set theory, "namely the problem of semantics of fuzzy sets (that is, how to ensure that the fuzzy sets used really represent the meaning that people attach to them), and secondly the methodological aspects of fuzzy set theory, (that is, what the methodology of science has to say about this new, rapidly advancing theory)."

Fuzzy sets use a closed scaling system with values that range between [0,1] to describe gradual membership functions. The assignment of any value or evaluation level of membership is exposed to subjective judgment. This classical problem, though, becomes minimized because one of the strengths with fuzzy set theory is the emphasis that objectives and constraints can be represented by relationships which subsume elements of subjective preference. Fuzziness in assignment is okay, and it is natural. Fuzzy theory frees up the need of applying the classical constraint of exactness to fit rigid categorical assignments.

Mathematics of fuzzy sets is not "logical" from the perspective of the rational theoretical models, and it is not subjectable to the optimitizing and maximizing assumptions in micro-economics; fuzzy modeling is not applicable to Pareto optimality. Fuzzy optimization does not purport to distinguish the best solution but seeks a good solution. When caught up in a world driven by concepts with perceived needs (or mandated requirements) to increase efficiency and productivity and lower costs by fine tuning the processes used to convert resources to outputs, the use of fuzzy theory could be an unsettling notion. Efficiency and productivity may have to give way to looser fitting criteria and allow more tolerance in a fuzzy perspective.

It has been said that fuzzy theory and its mathematical applications are not as easily applied as the more commonly used concepts from general linear models based on means, standard deviations, and standard errors. But this problem is possibly a matter of reflection on the predisposition of having institutionalized the framing of issues in an Aristotelian style of logic. Changing the frame of reference may make the application of fuzzy set theory to problems just as easy as using the concepts in general linear models.

Fuzzy Acceptance Theory

Bernstein (1976) stated, "an adequate comprehensive political and social theory must at once be empirical, interpretative, and critical." Can a fuzzy set theory orientation provide such comprehensiveness? The premise for fuzzy acceptance theory to be endorsed by public managers rests in their acceptance of (a) possibility measurement, (b) use of graded membership in sets to indicate levels of certainty, (c) an application to which events are interpreted as deterministic, and (d) the fuzzy set collaries (see Appendix A) are operative.

Kosko's (1989) n-dimensional hypercube is the model that can start to bring together what appears to be differing points of view or disconnected perspectives that have in the past been referred to as a process of dialectical conflict between thesis and antithesis. Facts and values do not have to be dichotomous, nor stored in mutually exclusive categories. The fuzzy acceptance theory metaphor implies the thesis, antithesis, synthesis, old paradigm and new paradigm co-exist where each entity is accountable for presenting, describing, and contributing the facts and values within their unique scope. If old, or antiquated, ideas cannot co-exist with the new, the future can fall victim of recreating the past. Just because the model to live by socialism as created in the Soviet Union's version of communism has dissolved does not dismiss the ideology.

The fuzzy acceptance theory can be used to understand theoretical constructs such as organizations, democracy, voting, public and private, effectiveness, and efficiency to name but a few of the primary public administrative constructs.

Gareth Morgan (1983) commented in his book, *Beyond Method*, that "social scientists deal with possibilities. They are concerned with the realization of possible knowledge, since what is studied and what is learned are intimately connected with the mode of engagement adopted." The mode of engagement Morgan referred to is any of the many modes of research as practiced among social scientists. To say any one engagement is better than the other requires an independent point of reference against which the nature and claims of the different research strategies can be assessed. Morgan argued that such a reference point does not exist.

> It is fallacious to conclude that the propositions of a system of thought can be proved, disproved, or evaluated on the basis of axioms within that system, since the process becomes self-justifying. Which means it is not possible to determine the validity or contribution of different research strategies in any absolute sense in terms of evaluative stances that draw on the same assumptions as do any of the research strategies examined.

An underlying property of fuzzy set theory is the interpretation of events as being deterministic, not random, whose fuzzy nature of measuring is of possibilities that represent certainty. A fuzzy acceptance theory orientation could provide Morgan's (1983) "independent point of reference necessary to avoid the problem of self-justifying claims." The fuzzy acceptance theory position would not pit one engagement against another in duals to be labeled the "right" research technique or the "right" theory, but instead share how each research technique has power to partially explain phenomenon. Cumulatively more can be explained and understood. That which is left unexplained, or in a confused state, is an indicator of the need for more n-dimensions to be established.

What we touch is the center of earnest effort as Morgan (1983) said, "The whole history of epistemology can be interpreted as hinging on the quest for certainty in our way of knowing."

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Appendix A
Fuzzy Set Properties

The algebraic properties—commutativity, associativity, distributivity and complimentation—of fuzzy sets are the same as for ordinary sets (i.e., binary sets with the two elements "0" and "1") using Boolean logic. The rules for intersection and union and compliment are the same. The primary difference in properties between ordinary sets logic and fuzzy sets logic are the properties of noncontradiction and exclusion.

- Commutativity: $A \cap B = B \cap A$
  $A \cup B = B \cup A$
- Associativity: $(A \cap B) \cap C = A \cap (B \cap C)$
  $(A \cup B) \cup C = A \cup (B \cup C)$
- Distributivity: $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
  $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
- The universe of elements in fuzzy set theory is any number in a closed interval from 0 to 1.
- A fuzzy set has a membership function with not only values of 0 (does not belong to) or 1 (belongs to), but any number in the interval 0 and 1 (For example, 0.3, 0.651, 0.98...that represent grades of membership.)
- The following two fuzzy sets (A,B) are used to illustrate.

$$A = \{.7\ \ .4\ \ 0\ \ .5\ \ .2\ \ 1\}$$
$$B = \{.3\ \ 1\ \ .4\ \ .9\ \ 0\ \ 1\}$$

- Intersection of A and B uses the minimum rule of selecting the smaller of the two elements

$$A \cap B = \{.3\ \ .4\ \ 0\ \ .5\ \ 0\ \ 1\}$$

- Union of A and B uses the maximum rule of selecting the larger of the two elements

$$A \cup B = \{.7\ \ 1\ \ .4\ \ .9\ \ .2\ \ 1\}$$

- The compliment of A (not A) is 1 minus the member in A:

$$A^c = \{.3\ \ .6\ \ 1\ \ .5\ \ .8\ \ 0\}$$

- Kosko (1989) reported "fuzziness arises from the ambiguity between a thing A and its opposite A^c (not A)." If we do not know A with certainty, we do not know A^c with certainty either. Else by double negation we would know A with certainty. This produces nondegenerate overlap: $A \cap A^c$ does not equal the null set, which breaks the "law of noncontradiction." Equivalently, this also produces nondegenerate underlap: $A \cup A^c$ does not equal X, which breaks the "law of excluded middle." Here X is the ground set or universe of discourse. These laws are never broken in probabilistic or stochastic logics—$P(A \text{ and } A^c) = 0$ and $P(A \text{ or } A^c) = 1$—even though they are broken with many, perhaps most, human utterances.
- The intersection of A and its compliment is not necessarily the null set as defined in classical set theory. Using the minimum rule the intersection is

$$A \cap A^c = \{.3\ \ .4\ \ 0\ \ .5\ \ .2\ \ 0\}$$

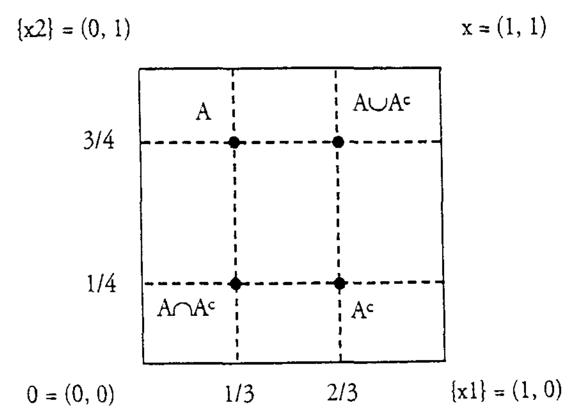
- The union of A and its compliment is not necessarily the empty set as defined in classical set theory. Using the maximum rule the union is

$$A \cup A^c = \{.7\ \ .6\ \ 1\ \ .5\ \ .8\ \ 1\}$$

Appendix B
Geometry of Fuzzy Sets: Sets as Points

Kosko's (1989) geometry of fuzzy sets as sets-of-points in a fuzzy square is presented. The geometry of fuzzy sets involves both the domain $X = \{x_1,...x_2\}$ and the range [0,1]. A fuzzy set is a point in a cube. In the diagram below, the fuzzy subset A is a point in the unit 2-cube with coordinates or fit values (1/3 3/4). The first element x_1 fits in or belongs to A to degree 1/3, the element x_2 to degree 3/4. The cube consists of all possible fuzzy subsets of two elements $\{x_1,x_2\}$. The four corners represent the power set 2x of $\{x_1,x_2\}$.

Proposition: A is properly fuzzy if and only if $A \cap Ac = 0$ and if and only if $A \cup A^c = X$.



An illustration of this fundamental proposition is what we might call completing the fuzzy square. Consider the two-dimensional fuzzy set A defined to the fit vector (1/3 3/4). The corresponding overlap and underlap sets can be found by first finding the compliment set A^c and then combining the fit vectors pairwise with minimum and maximum rules:

A = (1/3 3/4)
A^c = (2/3 1/4)
$A \cap A^c$ = (1/3 1/4)
$A \cup A^c$ = (2/3 3/4)

The sets-as-points view shows that these four points in the unit square hang together, indeed move together, in a very natural way as illustrated in the above diagram. The fuzzier A is the closer A is to the midpoint of the fuzzy cube. As A approaches the mid-point, all four points—A, A^c, $A \cap A^c$, $A \cup A^c$—contract to the midpoint. The less fuzzy A is, the closer A is to the nearest vertex. As A approaches a vertex, all four points spread out to the four vertices.

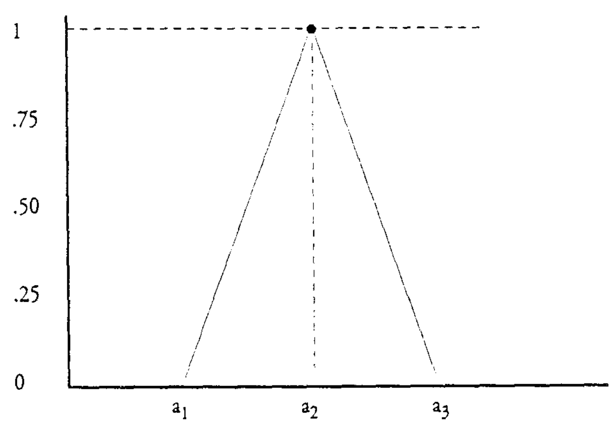
Appendix C
Triangular Fuzzy Numbers (TFNs)

This appendix is not the definitive treatment of Triangular Fuzzy Numbers (TFN), but it is a review of the most important properties of these numbers in support of the discussion in this paper. Refer to Kaufmann and Gupta (1988) for detailed treatment of which the following is excerpted.

A special class of fuzzy numbers called triangular fuzzy numbers is illustrated in the figure below.

The Y axis indicates the level of membership (whose range is the fuzzy set [0,1]). The X-axis represents what has been measured. Triangular fuzzy numbers are defined as a triplet (a_1,a_2,a_3) within an interval of confidence and has a membership function defined by four parameters:

Figure C1
A Triangular Fuzzy Number (TFN) $A = (a_1, a_2, a_3)$.



uA(x) is a membership function for the element x with respect to the fuzzy subset A.

uA(x) = 0 when $x < a_1$
uA(x) = $x-a_1/a_2-a_1$ when $a_1 <= x <= a_2$
uA(x) = a_3-x/a_3-a_2 when $a_2 <= x <= a_3$
uA(x) = 0 when $x > a_3$.

The interval of confidence represents differing levels of certainty—another often used term is possibility. The interval of confidence at level alpha follows:

$A = [(a_2-a_1)\alpha + a_1, - (a_3-a_2)\alpha + a_3]$.

The interval of confidence at alpha designates the amount of membership along either the increasing or decreasing slopes in the figure above. When alpha = 1, then the membership function is at most upper point at a_2, while when alpha = 0 the membership function can be either at a1 or a_3.

In passing, another useful class of fuzzy numbers is the trapezoidal fuzzy number (TrFN) that can be represented by $A=(a_1,a_2,a_3,a_4)$. In the case of TrFn, when the membership function of alpha = 1, rather than being a point as in TFNs, there is a flat line interval (a_2,a_3) while at a_1 and a_4 (the lower and upper thresholds) are equal to zero. The four points form a trapezoid area when plotted. TrFN hold similar properties as defined by TFNs and the mathematics involved are similar.